LOVÁSZ THETA AND SHEARER LOWER BOUNDS ON QUANTUM MAX CUT

Felix Huber

University of Gdańsk

e-mail: felix.huber@ug.edu.pl

Quantum Max Cut is a problem relevant to computer science and many-body quantum physics due to its links to classical Max Cut and the anti-ferromagnetic Heisenberg Hamiltonian. *Quantum Max Cut* (QMC) is a generalization of Max Cut to operators, and asks for the largest eigenvalue of the operator

$$H^{\mathrm{qmc}} = \frac{1}{4} \sum_{uv \in E} (I - X_u X_v - Y_u Y_v - Z_u Z_v)\,.$$

Here I is the identity matrix and X_u, Y_u, Z_u are the tensor product operators acting with the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on vertex u and with identity on the remaining vertices. Thus H acts on $(\mathbb{C}^2)^{\otimes |V|}$. Denote by $\mathrm{qmc}(G)$ the largest eigenvalue of H^{qmc}.

We prove a lower bound to quantum Max Cut of a graph in terms of the Lovász theta function of its complement. For a graph with m edges,

$$\mathrm{qmc}(G) \geq \frac{m}{4}\Big(1 + \frac{8}{3\pi}\frac{1}{\vartheta(\bar{G}) - 1}\Big)\,,$$

with the bound achieved by a product state. The proof extends a result by Balla, Janzer, and Sudakov on classical Max Cut [1], and can be strenghtened by the vector chromatic number. A relaxed bound follows from $\vartheta(\bar{G}) - 1 \leq \Delta$ for graphs with maximum degree Δ, making it interesting for practically relevant quantum many-body systems.

We also extend results by Carlson et al. [2] and Shearer [3] and show that

$$\mathrm{qmc}(G) \geq \frac{m}{4} + \frac{2m^{3/4}}{3\pi}$$

for all triangle-free graphs with m edges.

References

[1] Igor Balla, Oliver Janzer, and Benny Sudakov. On MaxCut and the Lovász theta function. Proceedings of the American Mathematical Society, 152:1871–1879, 2024

[2] Charles Carlson, Alexandra Kolla, Ray Li, Nitya Mani, Benny Sudakov, and Luca Trevisan. Lower bounds for max-cut in H-free graphs via semidefinite programming, 2020. arXiv:1810.10044

[3] James B. Shearer. A note on bipartite subgraphs of triangle-free graphs. Random Structures & Algorithms, 3(2):223–226, 1992.