

DOMINATION MODELS IN FUZZY AND WEIGHTED GRAPHS

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Fuzzy graphs represent a highly valuable tool for modeling and analyzing real-world scenarios, such as facility location problems. In this work, we utilize them to address the "15-minute walking city" problem, aiming to strategically place facilities within a strict maximum distance. However, computationally tackling these models can be challenging, as fuzzy relationships often impose rigid constraints on the graph's structure. To overcome these limitations and introduce greater structural flexibility, we also explore Roman domination models on weighted graphs, offering a more adaptable alternative for efficient problem-solving.

Let (G, w) be a vertex-weighted graph. We introduce the *weighted Roman domination number* $\gamma_{wR}(G)$ as the minimum weight of a Roman dominating function with respect to the vertex-weight function w . This parameter naturally extends the classical Roman domination number and provides a framework for studying Roman domination in heterogeneous networks.

We establish several fundamental properties of $\gamma_{wR}(G)$. First, we derive general bounds relating the weighted Roman domination number to the weighted domination number, the total weight of the graph, and the maximum weighted degree. In particular, we prove

$$\gamma_w(G) \leq \gamma_{wR}(G) \leq 2\gamma_w(G),$$

and characterize the weighted graphs attaining the extremal cases. We also obtain a sharp lower bound involving the maximum weighted degree and derive a Nordhaus–Gaddum-type inequality for the sum of the weighted Roman domination numbers of a graph and its complement.

The structure of minimum weighted Roman dominating functions is investigated in detail. Several structural properties of the corresponding vertex partition are established, leading to characterizations of weighted graphs satisfying $\gamma_{wR}(G) = w(G)$ and $\gamma_{wR}(G) = \gamma_w(G)$.

Furthermore, exact values of γ_{wR} are obtained for important graph families, including weighted complete graphs and weighted complete bipartite graphs. For weighted cycles, we derive sharp upper bounds and characterize the equality cases whenever the order of the cycle is not divisible by three.

Our main result establishes a direct connection between weighted Roman domination and the differential of a weighted graph. Specifically, we prove that

$$\gamma_{wR}(G) = w(G) - \partial(G),$$

where $\partial(G)$ denotes the weighted differential of (G, w) . This identity extends a classical relationship between domination-type parameters and graph differential, and allows techniques from both areas to be combined.

These results provide the first systematic study of Roman domination in weighted graphs and open several new directions for further research.

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