ON DEGREE-BALANCED EDGE PARTITIONS

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We investigate two notions of degree-balanced edge decompositions, which seemingly have nothing in common but surprisingly intertwine in various ways.

The first of these concerns majority edge decompositions, more commonly known as majority edge colourings. A graph is said to be majority edge c-colourable if its edges can be assigned one of c colours in such a way that each colour class induces a subgraph including at most $1/2$ the edges incident with every vertex. A natural generalization of this concept allows at most $1/k$'th fraction of monochromatic edges incident to any vertex. If one aims to use at most $k+1$ colours to achieve such an edge colouring, which is typically the least reasonable number, this usually results in nearly balanced degrees across different colours at each vertex. We particularly focus on list variants of this notion. An elegant result in this direction builds upon Galvin's well-known theorem related to Dinitz's conjecture.

The second problem concerns a conjecture that every d-regular graph G of order n contains a subgraph H which is called irregular. This subgraph is required to have a nearly balanced degree distribution, meaning that for any fixed degree, the number of vertices attaining it in H differs from $n/(d+1)$ by at most 2. We discuss a solution to this problem in the case of cubic graphs, for which an optimal result can be obtained – one even stronger than the postulated above.

This conjecture was formulated in 2023 by Alon and Wei in a paper where the authors introduced several interesting tools to take advantage of the probabilistic method in obtaining an asymptotic approximation of the desired property. Interestingly, one of these tools, derived from linear algebra, played a key role in the study of the first of the two discussed problems.

References

[1] N. Alon and F. Wei, Irregular subgraphs. Combinatorics, Probability and Computing 32(2) (2023), 269–283.

[2] F. Bock, R. Kalinowski, J. Pardey, M. Pilśniak, D. Rautenbach, and M. Woźniak, Majority Edge-Colorings of Graphs. Electron. J. Combin. 30(1) (2023), #P1.42.

[3] J. Fox, S. Luo, and H.T. Pham, On random irregular subgraphs. Random Struct. Alg. 64 (2024), 899–917.

[4] B. Lužar, J. Przybyło, and R. Soták, Degree-balanced decompositions of cubic graphs. European J. Comb. 128 (2025) 104169.

[5] J. Ma and S. Xie, Finding irregular subgraphs via local adjustments. Journal of Combinatorial Theory, Series B 174 (2025), 71–98.

[6] P. Pękała and J. Przybyło, On Generalised Majority Edge-Colourings of Graphs. Electron. J. Combin. 31(4) (2024), #P4.66.