DOMINATION NUMBER OF THE CARTESIAN PRODUCT OF A PATH WITH ANY PAIR OF GRAPHS

OMAR TOUT

Sultan Qaboos University

e-mail: o.tout@squ.edu.om

For a simple finite graph G, let $V(G)$ denote the set of vertices of G. We say that a vertex $u \in V(G)$ dominates a vertex v if $u = v$ or v is adjacent to u. A dominating set of G, is a subset of vertices of G which dominates all the vertices of G. The domination number of G, denoted $\gamma(G)$, is the size of a smallest dominating set of G. The Cartesian product $X \Box Y$ of two graphs X and Y is the graph whose vertex set is $V(X) \times V(Y)$ and edge set defined as follows. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $X \Box Y$ if either $x_1 = x_2$ and y_1 and y_2 are adjacent in Y, or $y_1 = y_2$ and x_1 and x_2 are adjacent in X. The still open conjecture of Vizing, see [1], states that $\gamma(X \Box Y) \geq \gamma(X)\gamma(Y)$ for any pair of graphs X and Y.

In 2000, Clark and Suen showed in [2] that $\gamma(X \Box Y) \geq \frac{1}{2}\gamma(X)\gamma(Y)$ for any pair of graphs X and Y. To this date, $\frac{1}{2}$ remains the best obtained coefficient towards proving Vizing's conjecture. Clark and Suen's result implies that $\gamma(X \Box Y \Box Z) \geq \frac{1}{4}\gamma(X)\gamma(Y)\gamma(Z)$ for any triple of graphs X, Y and Z. We show that this lower bound can be improved for special graphs. In particular, for any $n \geq 1$, we show that $\gamma(X \Box Y \Box P_n) \geq c_n \gamma(X)\gamma(Y)\gamma(P_n)$ where c_n is almost $\frac{3}{4}$ when n is big enough. Our proof can be found in [3]. It uses space projections and follows the new framework to approach Vizing's conjecture which appeared in [4].

References

[1] V. G. Vizing, *Some unsolved problems in graph theory*, Russian Math. Surveys 23(6) (1968) 125-141.

[2] W. E. Clark and S. Suen, *An inequality related to Vizing's conjecture*, Electronic Journal of Combinatorics 7 (2000) # N4.

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[4] B. Brešar, B. L. Hartnell, M. A. Henning, K. Kuenzel and D. F. Rall, *A new framework to approach Vizing's conjecture*, Discussiones Mathematicae Graph Theory. 41(3) (2021) 749-762.